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A

SHORT TABLE OF INTEGRALS.

COMPILED BY

B. O. PEIRCE,

HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY
IN HARVARD UNIVERSITY.



BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
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1893.

Math 30.68.95

1895, May 10.

Scientific School.
300.13

JUN 20 1917
TRANSFERRED TO
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notice of errors in these formulas to*

B. O. PEIRCE,

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I. FUNDAMENTAL FORMS.

1. $\int a \, dx = ax.$
2. $\int a f(x) \, dx = a \int f(x) \, dx.$
3. $\int \frac{dx}{x} = \log x.$
4. $\int x^m \, dx = \frac{x^{m+1}}{m+1},$ when m is different from $-1.$
5. $\int e^x \, dx = e^x.$
6. $\int a^x \log a \, dx = a^x.$
7. $\int \frac{dx}{1+x^2} = \tan^{-1} x.$
8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$
9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x.$
10. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1} x.$
11. $\int \cos x \, dx = \sin x.$
12. $\int \sin x \, dx = -\cos x.$

$$13. \int \operatorname{ctn} x dx = \log \sin x.$$

$$14. \int \tan x dx = -\log \cos x.$$

$$15. \int \tan x \sec x dx = \sec x.$$

$$16. \int \sec^2 x dx = \tan x.$$

$$17. \int \csc^2 x dx = -\operatorname{ctn} x.$$

In the following formulas, u , v , w , and y represent any functions of x :

$$18. \int (u + v + w + \text{etc.}) dx = \int u dx + \int v dx + \int w dx + \text{etc.}$$

$$19a. \int u dv = uv - \int v du.$$

$$19b. \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

$$20. \int f(y) dx = \int \frac{f(y) dy}{\frac{dy}{dx}}.$$

II. RATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $(a + bx)$.

The substitution of y or z for x , where $y = ax = a + bx$, gives

$$21. \int (a + bx)^m dx = \frac{1}{b} \int y^m dy.$$

$$22. \int x (a + bx)^m dx = \frac{1}{b^2} \int y^m (y - a) dy.$$

$$23. \int x^n (a + bx)^m dx = \frac{1}{b^{n+1}} \int y^m (y - a)^n dy.$$

$$24. \int \frac{x^n dx}{(a + bx)^m} = \frac{1}{b^{n+1}} \int \frac{(y - a)^n dy}{y^m}.$$

$$25. \int \frac{dx}{x^n (a + bx)^m} = -\frac{1}{a^{m+n-1}} \int \frac{(z - b)^{m+n-2} dz}{z^m}.$$

Whence

$$26. \int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx).$$

$$27. \int \frac{dx}{(a + bx)^2} = -\frac{1}{b (a + bx)}.$$

$$28. \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b (a + bx)^2}.$$

$$29. \int \frac{x dx}{a + bx} = \frac{1}{b^2} [a + bx - a \log (a + bx)].$$

$$30. \int \frac{x dx}{(a + bx)^2} = \frac{1}{b^2} \left[\log (a + bx) + \frac{a}{a + bx} \right].$$

$$31. \int \frac{x dx}{(a+bx)^3} = \frac{1}{b^2} \left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2} \right].$$

$$32. \int \frac{x^2 dx}{a+bx} = \frac{1}{b^3} \left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log(a+bx) \right].$$

$$33. \int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3} \left[a+bx - 2a \log(a+bx) - \frac{a^2}{a+bx} \right].$$

$$34. \int \frac{dx}{x(a+bx)} = -\frac{1}{a} \log \frac{a+bx}{x}.$$

$$35. \int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \log \frac{a+bx}{x}.$$

$$36. \int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a+bx}{x}.$$

B. — EXPRESSIONS INVOLVING $(a+bx^n)$.

$$37. \int \frac{dx}{c^2+x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}.$$

$$38. \int \frac{dx}{c^2-x^2} = \frac{1}{2c} \log \frac{c+x}{c-x}.$$

$$39. \int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} x \sqrt{\frac{b}{a}}, \text{ if } a > 0, b > 0.$$

$$40. \int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}}, \text{ if } a > 0, b < 0.$$

$$41. \int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a} \int \frac{dx}{a+bx^2}.$$

$$42. \int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma} \frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma} \int \frac{dx}{(a+bx^2)^m}.$$

$$43. \int \frac{x dx}{a+bx^2} = \frac{1}{2b} \log \left(x^2 + \frac{a}{b} \right).$$

$$44. \int \frac{x dx}{(a + bx^2)^{m+1}} = \frac{1}{2} \int \frac{dz}{(a + bz)^{m+1}}, \text{ where } z = x^2.$$

$$45. \int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x^2}{a + bx^2}.$$

$$46. \int \frac{x^2 dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}.$$

$$47. \int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}.$$

$$48. \int \frac{x^2 dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}.$$

$$49. \int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}.$$

$$50. \int \frac{dx}{a + bx^3} = \frac{k}{3a} \left[\frac{1}{2} \log \left(\frac{(k+x)^2}{k^2 - kx + x^2} \right) + \sqrt{3} \tan^{-1} \frac{2x - k}{k\sqrt{3}} \right],$$

where $bk^3 = a$.

$$51. \int \frac{x dx}{a + bx^3} = \frac{1}{3bk} \left[\frac{1}{2} \log \left(\frac{k^2 - kx + x^2}{(k+x)^2} \right) + \sqrt{3} \tan^{-1} \frac{2x - k}{k\sqrt{3}} \right],$$

where $bk^3 = a$.

$$52. \int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}.$$

$$53. \int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a + bx^n)^m} - \frac{b}{a} \int \frac{x^n dx}{(a + bx^n)^{m+1}}.$$

$$54. \int \frac{x^n dx}{(a + bx^n)^{p+1}} = \frac{1}{b} \int \frac{x^{n-p}}{(a + bx^n)^p} - \frac{a}{b} \int \frac{x^{n-p} dx}{(a + bx^n)^{p+1}}.$$

$$55. \int \frac{dx}{x^n(a + bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^n(a + bx^n)^p} - \frac{b}{a} \int \frac{dx}{x^{n-p}(a + bx^n)^{p+1}}.$$

$$56. \int x^{m-1}(a+bx^p)^p dx = \begin{cases} \frac{1}{b(m+np)} \left[x^{m-n}(a+bx^p)^{p+1} - (m-n)a \int x^{m-n-1}(a+bx^p)^p dx \right] \\ \frac{1}{m+np} \left[x^m(a+bx^p)^p + npa \int x^{m-1}(a+bx^p)^{p-1} dx \right] \\ \frac{1}{ma} \left[x^m(a+bx^p)^{p+1} - (m+np+n)b \int x^{m+n-1}(a+bx^p)^p dx \right] \\ \frac{1}{an(p+1)} \left[-x^m(a+bx^p)^{p+1} + (m+np+n) \int x^{m-1}(a+bx^p)^{p+1} dx \right] \end{cases}$$

C. — EXPRESSIONS INVOLVING $(a+bx+cx^2)$.

Let $X = a+bx+cx^2$ and $q = 4ac - b^2$, then

$$57. \int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx+b}{\sqrt{q}}, \text{ when } q > 0.$$

$$60. \int \frac{dx}{X^2} = \frac{2cx+b}{q} \left(\frac{1}{2X} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}.$$

$$58. \int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \text{ when } q < 0.$$

$$61. \int \frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n}.$$

$$59. \int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q} \int \frac{dx}{X}.$$

$$62. \int \frac{x dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$$

$$63. \int \frac{x dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.$$

$$64. \int \frac{x dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}.$$

$$65. \int \frac{x^2 dx}{X} = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2-2ac}{2c^2} \int \frac{dx}{X}.$$

$$66. \int \frac{x^2 dx}{X^2} = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}.$$

$$67. \int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}.$$

$$68. \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}.$$

$$69. \int \frac{dx}{x^2 X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}.$$

$$70. \int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} \\ - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}.$$

D. — RATIONAL FRACTIONS.

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)} = \frac{g_1 x^{n-1} + g_2 x^{n-2} + g_3 x^{n-3} + \dots + g_n}{x^n + k_1 x^{n-1} + k_2 x^{n-2} + \dots + k_n}.$$

a, b, c , etc., are the roots of the equation $F(x)=0$, so that

$$F(x) = (x-a)^p (x-b)^q (x-c)^r \dots,$$

$$\begin{aligned} \text{then } \frac{f(x)}{F(x)} = & \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \dots + \frac{A_p}{x-a} \\ & + \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \dots + \frac{B_q}{x-b} \\ & + \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \dots + \frac{C_r}{x-c} \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2^{[m-1]}(b)}{(m-1)!}, \text{ etc., etc.}$$

$$\phi_1(x) = \frac{f(x)(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)(x-b)^q}{F(x)}, \text{ etc., etc.}$$

If a, b, c , etc., are single roots, then $p = q = r = \dots = 1$, and

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \dots$$

where $A = \frac{f(a)}{F'(a)}, B = \frac{f(b)}{F'(b)}, \text{ etc.}$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the following formulas :

$$71. \int \frac{h dx}{(mx+n)^i} = \int \frac{h d(mx+n)}{m(mx+n)^i} = \frac{h}{m(1-i)(mx+n)^{i-1}}.$$

$$72. \int \frac{h dx}{mx+n} = \frac{h}{m} \log(mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined together and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities :

$$73. \log(x \pm yi) = \frac{1}{2} \log(x^2 + y^2) \pm i \tan^{-1} \frac{y}{x}.$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y = \sqrt{a+bx}$, gives

$$74. \int \sqrt{a+bx} dx = \frac{2}{3b} \sqrt{(a+bx)^3}.$$

$$75. \int x \sqrt{a+bx} dx = -\frac{2(2a-3bx) \sqrt{(a+bx)^3}}{15b^2}.$$

$$76. \int x^2 \sqrt{a+bx} dx = \frac{2(8a^2-12abx+15b^2x^2) \sqrt{(a+bx)^3}}{105b^3}.$$

$$77. \int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x \sqrt{a+bx}}.$$

$$78. \int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

$$79. \int \frac{x dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2} \sqrt{a+bx}.$$

$$80. \int \frac{x^2 dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3} \sqrt{a+bx}.$$

$$81. \int \frac{dx}{x \sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right), \text{ for } a > 0.$$

$$82. \int \frac{dx}{x \sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}, \text{ for } a < 0.$$

$$83. \int \frac{dx}{x^2 \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x \sqrt{a+bx}}.$$

$$84. \int (a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b} \int y^{1 \pm n} dy = \frac{2(a+bx)^{\frac{2 \pm n}{2}}}{b(2 \pm n)}.$$

$$85. \int x(a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b^2} \left[\frac{(a+bx)^{\frac{4 \pm n}{2}}}{4 \pm n} - \frac{a(a+bx)^{\frac{2 \pm n}{2}}}{2 \pm n} \right].$$

$$86. \int \frac{x^m dx}{\sqrt{a+bx}} = \frac{2x^m \sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b} \int \frac{x^{m-1} dx}{\sqrt{a+bx}}.$$

$$87. \int \frac{dx}{x^m \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1} \sqrt{a+bx}}.$$

$$88. \int \frac{(a+bx)^{\frac{n}{2}} dx}{x} = b \int (a+bx)^{\frac{n-2}{2}} dx + a \int \frac{(a+bx)^{\frac{n-2}{2}} dx}{x}.$$

$$89. \int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a} \int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a} \int \frac{dx}{(a+bx)^{\frac{m}{2}}}.$$

B. — EXPRESSIONS * INVOLVING $\sqrt{x^2 \pm a^2}$ AND $\sqrt{a^2 - x^2}$.

$$90. \int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x \sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})].$$

$$91. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).$$

$$92. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}).$$

$$93. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$$

$$94. \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \cos^{-1} \frac{a}{x}.$$

$$95. \int \frac{dx}{x \sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right).$$

$$96. \int \frac{\sqrt{a^2 \pm x^2}}{x} dx = \sqrt{a^2 \pm x^2} - a \log \frac{a + \sqrt{a^2 \pm x^2}}{x}.$$

* These equations are all special cases of more general equations given in the next section.

$$97. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.$$

$$98. \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2}.$$

$$99. \int \frac{x dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$$

$$100. \int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}.$$

$$101. \int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}.$$

$$102. \int \sqrt{(x^2 \pm a^2)^3} dx \\ = \frac{1}{4} \left[x \sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2 x}{2} \sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right].$$

$$103. \int \sqrt{(a^2 - x^2)^3} dx \\ = \frac{1}{4} \left[x \sqrt{(a^2 - x^2)^3} + \frac{3a^2 x}{2} \sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{a} \right].$$

$$104. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}.$$

$$105. \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}.$$

$$106. \int \frac{x dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$$

$$107. \int \frac{x dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

$$108. \int x \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}.$$

$$109. \int x \sqrt{(a^2 - x^2)^3} dx = -\frac{1}{5} \sqrt{(a^2 - x^2)^5}.$$

$$110. \int x^2 \sqrt{x^2 \pm a^2} dx \\ = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} (x \sqrt{x^2 \pm a^2} \pm a^2 \log (x + \sqrt{x^2 \pm a^2})).$$

$$111. \int x^2 \sqrt{a^2 - x^2} dx \\ = -\frac{x}{4} \sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).$$

$$112. \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$$

$$113. \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

$$114. \int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \pm \frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$$

$$115. \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

$$116. \int \frac{\sqrt{x^2 \pm a^2} dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log x + \sqrt{x^2 \pm a^2}.$$

$$117. \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}.$$

$$118. \int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log (x + \sqrt{x^2 \pm a^2}).$$

$$119. \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a}.$$

C. — EXPRESSIONS INVOLVING $\sqrt{a + bx + cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a + bx + cx^2})$ we may put $\sqrt{a + bx + cx^2} = \sqrt{\pm c} \sqrt{A + Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z , such that

$$z = \sqrt{A+Bx+x^2} - x, \text{ if } c > 0.$$

$$z = \frac{\sqrt{A+Bx-x^2}-\sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0.$$

$$z = \sqrt{\frac{x-\beta}{a-x}}, \text{ where } a \text{ and } \beta \text{ are the roots of the equation}$$

$$A+Bx-x^2=0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals :

$$120. \int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log \left(\sqrt{X} + x \sqrt{c} + \frac{b}{2\sqrt{c}} \right), \text{ if } c > 0.$$

$$121. \int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{-2cx-b}{\sqrt{b^2-4ac}} \right), \text{ if } c < 0.$$

$$122. \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$$

$$123. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}} \left(\frac{1}{X} + 2k \right).$$

$$124. \int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b\sqrt{X})}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}}.$$

$$125. \int \sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}.$$

$$126. \int X\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{8c} \left(X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}.$$

$$127. \int X^2\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{12c} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}.$$

$$128. \int X^n\sqrt{X} dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int \frac{X^n dx}{\sqrt{X}}.$$

$$129. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}.$$

$$56. \int x^{m-1}(a+bx^n)^p dx = \left\{ \begin{array}{l} \frac{1}{b(m+np)} \left[x^{m-n}(a+bx^n)^{p+1} - (m-n)a \int x^{m-n-1}(a+bx^n)^p dx \right], \\ \frac{1}{m+np} \left[x^m(a+bx^n)^p + npa \int x^{m-1}(a+bx^n)^{p-1} dx \right], \\ \frac{1}{ma} \left[x^m(a+bx^n)^{p+1} - (m+np+n)b \int x^{m+n-1}(a+bx^n)^p dx \right], \\ \frac{1}{an(p+1)} \left[-x^m(a+bx^n)^{p+1} + (m+np+n) \int x^{m-1}(a+bx^n)^{p+1} dx \right]. \end{array} \right.$$

C.—EXPRESSIONS INVOLVING $(a+bx+cx^2)$.

Let $X = a+bx+cx^2$ and $q = 4ac - b^2$, then

$$57. \int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx+b}{\sqrt{q}}, \text{ when } q > 0. \quad 60. \int \frac{dx}{X^2} = \frac{2cx+b}{q} \left(\frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}.$$

$$58. \int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \text{ when } q < 0. \quad 61. \int \frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n}.$$

$$59. \int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q} \int \frac{dx}{X}. \quad 62. \int \frac{x dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$$

$$63. \int \frac{x dx}{X^2} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.$$

$$64. \int \frac{x dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}.$$

$$65. \int \frac{x^2 dx}{X} = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}.$$

$$66. \int \frac{x^3 dx}{X^2} = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}.$$

$$67. \int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}.$$

$$68. \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}.$$

$$69. \int \frac{dx}{x^2 X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}.$$

$$70. \int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} \\ - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}.$$

D. — RATIONAL FRACTIONS.

Every proper fraction can be represented by the general form :

$$\frac{f(x)}{F(x)} = \frac{g_1 x^{n-1} + g_2 x^{n-2} + g_3 x^{n-3} + \dots + g_n}{x^n + k_1 x^{n-1} + k_2 x^{n-2} + \dots + k_n}.$$

a, b, c , etc., are the roots of the equation $F(x) = 0$, so that

$$F(x) = (x-a)^p (x-b)^q (x-c)^r \dots,$$

$$\begin{aligned} \text{then } \frac{f(x)}{F(x)} = & \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \dots + \frac{A_p}{x-a} \\ & + \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \dots + \frac{B_q}{x-b} \\ & + \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \dots + \frac{C_r}{x-c} \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2^{[m-1]}(b)}{(m-1)!}, \text{ etc., etc.}$$

$$\phi_1(x) = \frac{f(x)(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)(x-b)^q}{F(x)}, \text{ etc., etc.}$$

If $a, b, c, \text{ etc.}$, are single roots, then $p = q = r = \dots = 1$, and

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \dots$$

where $A = \frac{f(a)}{F'(a)}, B = \frac{f(b)}{F'(b)}, \text{ etc.}$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the following formulas:

$$71. \int \frac{h dx}{(mx+n)^i} = \int \frac{h d(mx+n)}{m (mx+n)^i} = \frac{h}{m(1-i)(mx+n)^{i-1}}.$$

$$72. \int \frac{h dx}{mx+n} = \frac{h}{m} \log (mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined together and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

$$73. \log(x \pm yi) = \frac{1}{2} \log(x^2 + y^2) \pm i \tan^{-1} \frac{y}{x}.$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y = \sqrt{a+bx}$, gives

$$74. \int \sqrt{a+bx} \, dx = \frac{2}{3b} \sqrt{(a+bx)^3}.$$

$$75. \int x \sqrt{a+bx} \, dx = -\frac{2(2a-3bx) \sqrt{(a+bx)^3}}{15b^2}.$$

$$76. \int x^2 \sqrt{a+bx} \, dx = \frac{2(8a^2-12abx+15b^2x^2) \sqrt{(a+bx)^3}}{105b^3}.$$

$$77. \int \frac{\sqrt{a+bx}}{x} \, dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}}.$$

$$78. \int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

$$79. \int \frac{x \, dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2} \sqrt{a+bx}.$$

$$80. \int \frac{x^2 \, dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3} \sqrt{a+bx}.$$

$$81. \int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right), \text{ for } a > 0.$$

$$82. \int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}, \text{ for } a < 0.$$

$$83. \int \frac{dx}{x^2 \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{a+bx}}.$$

$$84. \int (a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b} \int y^{1 \pm n} dy = \frac{2(a+bx)^{\frac{2 \pm n}{2}}}{b(2 \pm n)}.$$

$$85. \int x(a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b^2} \left[\frac{(a+bx)^{\frac{4 \pm n}{2}}}{4 \pm n} - \frac{a(a+bx)^{\frac{2 \pm n}{2}}}{2 \pm n} \right].$$

$$86. \int \frac{x^m dx}{\sqrt{a+bx}} = \frac{2x^m \sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b} \int \frac{x^{m-1} dx}{\sqrt{a+bx}}.$$

$$87. \int \frac{dx}{x^m \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1} \sqrt{a+bx}}.$$

$$88. \int \frac{(a+bx)^{\frac{n}{2}} dx}{x} = b \int (a+bx)^{\frac{n-2}{2}} dx + a \int \frac{(a+bx)^{\frac{n-2}{2}}}{x} dx.$$

$$89. \int \frac{dx}{x(a+bx)^{\frac{n}{2}}} = \frac{1}{a} \int \frac{dx}{x(a+bx)^{\frac{n-2}{2}}} - \frac{b}{a} \int \frac{dx}{(a+bx)^{\frac{n}{2}}}.$$

B. — EXPRESSIONS * INVOLVING $\sqrt{x^2 \pm a^2}$ AND $\sqrt{a^2 - x^2}$.

$$90. \int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})].$$

$$91. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).$$

$$92. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}).$$

$$93. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$$

$$94. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \cos^{-1} \frac{a}{x}.$$

$$95. \int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right).$$

$$96. \int \frac{\sqrt{a^2 \pm x^2}}{x} dx = \sqrt{a^2 \pm x^2} - a \log \frac{a + \sqrt{a^2 \pm x^2}}{x}.$$

* These equations are all special cases of more general equations given in the next section.

$$97. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.$$

$$98. \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2}.$$

$$99. \int \frac{x dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$$

$$100. \int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}.$$

$$101. \int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}.$$

$$102. \int \sqrt{(x^2 \pm a^2)^3} dx \\ = \frac{1}{4} \left[x \sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2 x}{2} \sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right].$$

$$103. \int \sqrt{(a^2 - x^2)^3} dx \\ = \frac{1}{4} \left[x \sqrt{(a^2 - x^2)^3} + \frac{3a^2 x}{2} \sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{a} \right].$$

$$104. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}.$$

$$105. \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}.$$

$$106. \int \frac{x dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$$

$$107. \int \frac{x dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

$$108. \int x \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}.$$

$$109. \int x \sqrt{(a^2 - x^2)^3} dx = -\frac{1}{5} \sqrt{(a^2 - x^2)^5}.$$

$$110. \int x^2 \sqrt{x^2 \pm a^2} dx \\ = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} (x \sqrt{x^2 \pm a^2} \pm a^2 \log (x + \sqrt{x^2 \pm a^2})).$$

$$111. \int x^2 \sqrt{a^2 - x^2} dx \\ = -\frac{x}{4} \sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).$$

$$112. \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$$

$$113. \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

$$114. \int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \pm \frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$$

$$115. \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

$$116. \int \frac{\sqrt{x^2 \pm a^2} dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log x + \sqrt{x^2 \pm a^2}.$$

$$117. \int \frac{\sqrt{a^2 - x^2} dx}{x^2} = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}.$$

$$118. \int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log (x + \sqrt{x^2 \pm a^2}).$$

$$119. \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a}.$$

C. — EXPRESSIONS INVOLVING $\sqrt{a + bx + cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a + bx + cx^2})$ we may put $\sqrt{a + bx + cx^2} = \sqrt{\pm c \sqrt{A + Bx \pm x^2}}$, according as c is positive or negative, and then substitute for x a new variable z , such that

$$z = \sqrt{A+Bx+x^3} - x, \text{ if } c > 0.$$

$$z = \frac{\sqrt{A+Bx-x^3}-\sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0.$$

$$z = \sqrt{\frac{x-\beta}{a-x}}, \text{ where } a \text{ and } \beta \text{ are the roots of the equation}$$

$$A+Bx-x^3=0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

$$120. \int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log \left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}} \right), \text{ if } c > 0.$$

$$121. \int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{-2cx-b}{\sqrt{b^2-4ac}} \right), \text{ if } c < 0.$$

$$122. \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$$

$$123. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}} \left(\frac{1}{X} + 2k \right).$$

$$124. \int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b\sqrt{X})}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}}.$$

$$125. \int \sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}.$$

$$126. \int X\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{8c} \left(X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}.$$

$$127. \int X^2\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{12c} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}.$$

$$128. \int X^n\sqrt{X} dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int \frac{X^n dx}{\sqrt{X}}.$$

$$129. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}.$$

$$130. \int \frac{x dx}{X \sqrt{X}} = -\frac{2(bx + 2a)}{q \sqrt{X}}.$$

$$131. \int \frac{x dx}{X^n \sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)c X^n} - \frac{b}{2c} \int \frac{dx}{X^n \sqrt{X}}.$$

$$132. \int \frac{x^2 dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right) \sqrt{X} + \frac{3b^2 - 4ac}{8c^2} \int \frac{dx}{\sqrt{X}}.$$

$$133. \int \frac{x^2 dx}{X \sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{cq \sqrt{X}} + \frac{1}{c} \int \frac{dx}{\sqrt{X}}.$$

$$134. \int \frac{x^2 dx}{X^n \sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{(2n-1)cq X^{n-1} \sqrt{X}} + \frac{4ac + (2n-3)b^2}{(2n-1)cq} \int \frac{dx}{X^{n-1} \sqrt{X}}.$$

$$135. \int \frac{x^3 dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^2} - \frac{2a}{3c^2}\right) \sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^2}\right) \int \frac{dx}{\sqrt{X}}.$$

$$136. \int x \sqrt{X} dx = \frac{X \sqrt{X}}{3c} - \frac{b}{2c} \int \sqrt{X} dx.$$

$$137. \int x X \sqrt{X} dx = \frac{X^2 \sqrt{X}}{5c} - \frac{b}{2c} \int X \sqrt{X} dx.$$

$$138. \int \frac{x X^n dx}{\sqrt{X}} = \frac{X^n \sqrt{X}}{(2n+1)c} - \frac{b}{2c} \int \frac{X^n dx}{\sqrt{X}}.$$

$$139. \int x^2 \sqrt{X} dx = \left(x - \frac{5b}{6c}\right) \frac{X \sqrt{X}}{4c} + \frac{5b^2 - 4ac}{16c^2} \int \sqrt{X} dx.$$

$$140. \int \frac{x^2 X^n dx}{\sqrt{X}} = \frac{x X^n \sqrt{X}}{2(n+1)c} - \frac{(2n+3)b}{4(n+1)c} \int \frac{x X^n dx}{\sqrt{X}} \\ - \frac{a}{2(n+1)c} \int \frac{X^n dx}{\sqrt{X}}.$$

$$141. \int x^2 \sqrt{X} dx = \left(x^2 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c}\right) \frac{X \sqrt{X}}{5c} \\ + \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^2}\right) \int \sqrt{X} dx.$$

$$142. \int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{X} + \sqrt{a}}{x} + \frac{b}{2\sqrt{a}} \right), \text{ if } a > 0.$$

$$143. \int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{bx + 2a}{x\sqrt{b^2 - 4ac}} \right), \text{ if } a < 0.$$

$$144. \int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \text{ if } a = 0.$$

$$145. \int \frac{dx}{xX^n\sqrt{X}} = \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a} \int \frac{dx}{X^n\sqrt{X}}.$$

$$146. \int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{X}}.$$

$$147. \int \frac{\sqrt{X} dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}.$$

$$148. \int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a \int \frac{X^{n-1} dx}{x\sqrt{X}} + \frac{b}{2} \int \frac{X^{n-1} dx}{\sqrt{X}}.$$

$$149. \int \frac{\sqrt{X} dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{X}} + c \int \frac{dx}{\sqrt{X}}.$$

$$150. \int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c} \int \frac{x^{m-2} dx}{X^{n-1}\sqrt{X}} - \frac{b}{c} \int \frac{x^{m-1} dx}{X^n\sqrt{X}} - \frac{a}{c} \int \frac{x^{m-2} dx}{X^n\sqrt{X}}.$$

$$151. \int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1} X^n \sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)} \int \frac{x^{m-1} X^n dx}{\sqrt{X}} \\ - \frac{(m-1)a}{(2n+m)c} \int \frac{x^{m-2} X^n dx}{\sqrt{X}}.$$

$$152. \int \frac{dx}{x^m X^n \sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n} \\ - \frac{(2n+2m-3)b}{2a(m-1)} \int \frac{dx}{x^{m-1}X^n\sqrt{X}} - \frac{(2n+m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

$$153. \int \frac{X^n dx}{x^m \sqrt{X}} = -\frac{X^{n-1} \sqrt{X}}{(m-1)x^{m-1}} + \frac{(2n-1)b}{2(m-1)} \int \frac{X^{n-1} dx}{x^{m-1} \sqrt{X}} \\ + \frac{(2n-1)c}{m-1} \int \frac{X^{n-1} dx}{x^{m-2} \sqrt{X}}.$$

D. — MISCELLANEOUS EXPRESSIONS.

$$154. \int \sqrt{2ax - x^2} dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{a}.$$

$$155. \int \frac{dx}{\sqrt{2ax - x^2}} = \text{versin}^{-1} \frac{x}{a}.$$

$$156. \int \frac{dx}{(x+1)\sqrt{x^2-1}} = +\sqrt{\frac{x-1}{x+1}}.$$

$$157. \int \frac{dx}{(x-1)\sqrt{x^2-1}} = -\sqrt{\frac{x+1}{x-1}}.$$

$$158. \int \sqrt{\frac{1+x}{1-x}} dx = \sin^{-1} x - \sqrt{1-x^2}.$$

$$159. \int \sqrt{\frac{x+a}{x+b}} dx = \sqrt{(x+a)(x+b)} \\ + (a-b) \log(\sqrt{x+a} + \sqrt{x+b}).$$

$$160. \int \frac{dx}{\sqrt{(x-a)(\beta-x)}} = 2 \sin^{-1} \sqrt{\frac{x-a}{\beta-a}}.$$

$$161. \int \frac{dx}{\sqrt{(a+bx)(a-\beta x)}} = \frac{2}{\sqrt{b\beta}} \sin^{-1} \sqrt{\frac{\beta(a+bx)}{a\beta+ba}}.$$

$$162. \int \sqrt[3]{a+bx} dx = \frac{3}{4b} \sqrt[3]{(a+bx)^4}.$$

$$163. \int \frac{dx}{\sqrt[3]{a+bx}} = \frac{3}{2b} \sqrt[3]{(a+bx)^2}.$$

$$164. \int \frac{x dx}{\sqrt[3]{a+bx}} = -\frac{3(3a-2bx)}{10b^2} \sqrt[3]{(a+bx)^2}.$$

$$165. \int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an} \sec^{-1}\left(\frac{x^n}{a}\right).$$

$$166. \int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an} \log \frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$$

IV. TRANSCENDENTAL FUNCTIONS.

$$167. \int \sin x dx = -\cos x.$$

$$168. \int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x.$$

$$169. \int \sin^3 x dx = -\frac{1}{2} \cos x (\sin^2 x + 2).$$

$$170. \int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx.$$

$$171. \int \cos x dx = \sin x.$$

$$172. \int \cos^2 x dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$173. \int \cos^3 x dx = \frac{1}{2} \sin x (\cos^2 x + 2).$$

$$174. \int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

$$175. \int \sin x \cos x dx = \frac{1}{2} \sin^2 x.$$

$$176. \int \sin^2 x \cos^2 x dx = -\frac{1}{8} \left(\frac{1}{2} \sin 4x - x \right).$$

$$177. \int \sin x \cos^m x dx = -\frac{\cos^{m+1} x}{m+1}.$$

$$178. \int \sin^n x \cos x dx = \frac{\sin^{n+1} x}{m+1}.$$

$$179. \int \cos^m x \sin^n x dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} \\ + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x dx.$$

$$180. \int \cos^m x \sin^n x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} \\ + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x dx.$$

$$181. \int \frac{\cos^m x dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1) \sin^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\cos^m x dx}{\sin^{n-2} x}.$$

$$182. \int \frac{\cos^m x dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n) \sin^{n-1} x} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} x dx}{\sin^n x}.$$

$$183. \int \frac{\sin^m x dx}{\cos^n x} = -\int \frac{\cos^m \left(\frac{\pi}{2} - x \right) d \left(\frac{\pi}{2} - x \right)}{\sin^n \left(\frac{\pi}{2} - x \right)}.$$

$$\begin{aligned} 184. \int \frac{dx}{\sin^m x \cos^n x} \\ &= \frac{1}{n-1} \cdot \frac{1}{\sin^{n-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cdot \cos^{n-2} x} \\ &= -\frac{1}{m-1} \cdot \frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cdot \cos^n x}. \end{aligned}$$

$$185. \int \frac{dx}{\sin^m x} = -\frac{1}{m-1} \cdot \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} x}.$$

$$186. \int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$$

$$187. \int \tan x dx = -\log \cos x.$$

$$188. \int \tan^2 x dx = \tan x - x.$$

$$189. \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

$$190. \int \operatorname{ctn} x dx = \log \sin x.$$

$$191. \int \operatorname{ctn}^2 x dx = -\operatorname{ctn} x - x.$$

$$192. \int \operatorname{ctn}^n x dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x dx.$$

$$193. \int \sec x dx = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$194. \int \sec^2 x dx = \tan x.$$

$$195. \int \sec^n x dx = \int \frac{dx}{\cos^n x}.$$

$$196. \int \csc x dx = \log \tan \frac{1}{2} x.$$

$$197. \int \csc^2 x dx = -\cot x.$$

$$198. \int \csc^n x dx = \int \frac{dx}{\sin^n x}.$$

$$199. \int \frac{dx}{a + b \cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1} \left[\frac{b + a \cos x}{a + b \cos x} \right],$$

$$\text{or} \quad \frac{1}{\sqrt{b^2 - a^2}} \log \left[\frac{b + a \cos x + \sqrt{b^2 - a^2} \cdot \sin x}{a + b \cos x} \right].$$

$$200. \int \frac{dx}{a + b \cos x + c \sin x}$$

$$= \frac{-1}{\sqrt{a^2 - b^2 - c^2}} \cdot \sin^{-1} \left[\frac{b^2 + c^2 + a(b \cos x + c \sin x)}{\sqrt{b^2 + c^2}(a + b \cos x + c \sin x)} \right]$$

$$= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log$$

$$\left[\frac{b^2 + c^2 + a(b \cos x + c \sin x) + \sqrt{b^2 + c^2 - a^2}(b \sin x - c \cos x)}{\sqrt{b^2 + c^2}(a + b \cos x + c \sin x)} \right].$$

$$201. \int x \sin x dx = \sin x - x \cos x.$$

$$202. \int x^2 \sin x dx = 2x \sin x - (x^2 - 2) \cos x.$$

$$203. \int x^3 \sin x dx = (3x^2 - 6) \sin x - (x^3 - 6x) \cos x.$$

$$204. \int x^m \sin x dx = -x^m \cos x + m \int x^{m-1} \cos x dx.$$

$$205. \int x \cos x dx = \cos x + x \sin x.$$

$$206. \int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x.$$

$$207. \int x^3 \cos x dx = (3x^2 - 6) \cos x + (x^3 - 6x) \sin x.$$

$$208. \int x^m \cos x \, dx = x^m \sin x - m \int x^{m-1} \sin x \, dx.$$

$$209. \int \frac{\sin x}{x^m} \, dx = -\frac{1}{m-1} \cdot \frac{\sin x}{x^{m-1}} + \frac{1}{m-1} \int \frac{\cos x}{x^{m-1}} \, dx.$$

$$210. \int \frac{\cos x}{x^m} \, dx = -\frac{1}{m-1} \cdot \frac{\cos x}{x^{m-1}} - \frac{1}{m-1} \int \frac{\sin x}{x^{m-1}} \, dx.$$

$$211. \int \frac{\sin x}{x} \, dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \dots$$

$$212. \int \frac{\cos x}{x} \, dx = \log x - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \frac{x^8}{8 \cdot 8!} \dots$$

$$218. \int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$$

$$214. \int \cos mx \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

$$215. \int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2}.$$

$$216. \int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1-x^2}.$$

$$217. \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2).$$

$$218. \int \operatorname{ctn}^{-1} x \, dx = x \operatorname{ctn}^{-1} x + \frac{1}{2} \log(1+x^2).$$

$$219. \int \operatorname{versin}^{-1} x \, dx = (x-1) \operatorname{versin}^{-1} x + \sqrt{2x-x^3}.$$

$$220. \int (\sin^{-1} x)^2 \, dx = x (\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2} \sin^{-1} x.$$

$$221. \int x \cdot \sin^{-1} x \, dx = \frac{1}{4} [(2x^2-1) \sin^{-1} x + x\sqrt{1-x^2}].$$

$$222. \int x^n \sin^{-1} x \, dx = \frac{x^{n+1} \sin^{-1} x}{n+1} - \frac{1}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1-x^2}}.$$

$$228. \int x^n \cos^{-1} x \, dx = \frac{x^{n+1} \cos^{-1} x}{n+1} + \frac{1}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1-x^2}}.$$

$$224. \int x^n \tan^{-1} x \, dx = \frac{x^{n+1} \tan^{-1} x}{n+1} - \frac{1}{n+1} \int \frac{x^{n+1} \, dx}{1+x^2}.$$

$$225. \int \log x \, dx = x \log x - x.$$

$$226. \int \frac{(\log x)^n}{x} \, dx = \frac{1}{n+1} (\log x)^{n+1}.$$

$$227. \int \frac{dx}{x \log x} = \log \cdot \log x.$$

$$228. \int \frac{dx}{x (\log x)^n} = -\frac{1}{(n-1) (\log x)^{n-1}}.$$

$$229. \int x^m \log x \, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right].$$

$$230. \int e^{ax} \, dx = \frac{e^{ax}}{a}.$$

$$231. \int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1).$$

$$232. \int x^m e^{ax} \, dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} \, dx.$$

$$233. \int \frac{e^{ax}}{x^m} \, dx = -\frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} \, dx.$$

$$234. \int e^{ax} \log x \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} \, dx.$$

$$235. \int e^{ax} \sin x \, dx = \frac{e^{ax} (a \sin x - \cos x)}{a^2 + 1}.$$

$$236. \int e^{ax} \cos x \, dx = \frac{e^{ax} (a \cos x + \sin x)}{a^2 + 1}.$$

DEFINITE INTEGRALS.

$$237. \int_0^{\infty} \frac{a dx}{a^2 + x^2} = \frac{\pi}{2}, \text{ if } a > 0; 0, \text{ if } a = 0; -\frac{\pi}{2}, \text{ if } a < 0.$$

$$238. \int_0^{\infty} x^{n-1} e^{-x} dx = \int_0^1 \left[\log \frac{1}{x} \right]^{n-1} dx = \Gamma(n).$$

$$\Gamma(n+1) = n \cdot \Gamma(n).$$

$$\Gamma(2) = \Gamma(1) = 1.$$

$$\Gamma(n+1) = n!, \text{ if } n \text{ is an integer}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$239. \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^{\infty} \frac{x^{m-1} dx}{(1+x)^{m+n}} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

$$240. \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (n-1)}{2 \cdot 4 \cdot 6 \dots (n)} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer.}$$

$$= \frac{2 \cdot 4 \cdot 6 \dots (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \dots n}, \text{ if } n \text{ is an odd integer.}$$

$$= \frac{1}{2} \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \text{ for any value of } n.$$

$$241. \int_0^{\infty} \frac{\sin mx dx}{x} = \frac{\pi}{2}, \text{ if } m > 0; 0, \text{ if } m = 0; -\frac{\pi}{2}, \text{ if } m < 0.$$

$$242. \int_0^{\infty} \frac{\sin x \cdot \cos mx dx}{x} = 0, \text{ if } m < -1 \text{ or } m > 1;$$

$$\frac{\pi}{4}, \text{ if } m = -1 \text{ or } m = 1; \frac{\pi}{2}, \text{ if } -1 < m < 1.$$

$$243. \int_0^{\infty} \frac{\sin^2 x dx}{x^2} = \frac{\pi}{2}.$$

$$244. \int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

$$245. \int_0^{\infty} \frac{\cos mx dx}{1+x^2} = \frac{\pi}{2} \cdot e^{-m}.$$

$$246. \int_0^{\infty} \frac{\cos x dx}{\sqrt{x}} = \int_0^{\infty} \frac{\sin x dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$$

$$247. \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} \\ = \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \dots \right], \text{ if } k^2 < 1. \\ = K.$$

$$248. \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 \sin^2 x} \cdot dx \\ = \frac{\pi}{2} \left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{k^6}{5} - \dots \right], \text{ if } k^2 < 1. \\ = E.$$

$$249. \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2a} \sqrt{\pi} = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right).$$

$$250. \int_0^{\infty} x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$$

$$251. \int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1.3.5 \dots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}.$$

$$252. \int_0^{\infty} e^{-x^2 - \frac{a^2}{x^2}} dx = \frac{e^{-2a} \sqrt{\pi}}{2}.$$

$$253. \int_0^{\infty} e^{-ax} \cos mx dx = \frac{a}{a^2 + m^2}, \text{ if } a > 0.$$

$$254. \int_0^{\infty} e^{-ax} \sin mx dx = \frac{m}{a^2 + m^2}, \text{ if } a > 0.$$

$$255. \int_0^{\infty} e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi} \cdot e^{-\frac{b^2}{4a^2}}}{2a}.$$

$$256. \int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}.$$

$$257. \int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12}.$$

$$258. \int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}.$$

$$259. \int_0^1 \log \left(\frac{1+x}{1-x} \right) \cdot \frac{dx}{x} = \frac{\pi^2}{4}.$$

$$260. \int_0^\infty \log \left(\frac{e^x+1}{e^x-1} \right) dx = \frac{\pi^2}{4}.$$

$$261. \int_0^1 \frac{dx}{\sqrt{\log \left(\frac{1}{x} \right)}} = \sqrt{\pi}.$$

$$262. \int_0^1 x^m \log \left(\frac{1}{x} \right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}.$$

$$263. \int_0^{\frac{\pi}{2}} \log \sin x dx = \int_0^{\frac{\pi}{2}} \log \cos x dx = -\frac{\pi}{2} \cdot \log 2.$$

$$264. \int_0^\pi x \cdot \log \sin x dx = -\frac{\pi^2}{2} \log 2.$$

AUXILIARY FORMULAS.

The following formulas are sometimes useful in the reduction of integrals :

$$265. \quad \log u = \log cu + \text{a constant.}$$

$$266. \quad \log(-u) = \log u + \text{a constant.}$$

$$267. \quad \sin^{-1} u = \begin{cases} -\sin^{-1} \sqrt{1-u^2} & + \text{a constant.} \\ -\frac{1}{2} \sin^{-1} (2u^2 - 1) & + \text{a constant.} \\ \frac{1}{2} \sin^{-1} 2u \sqrt{1-u^2} & + \text{a constant.} \end{cases}$$

$$268. \quad \tan^{-1} u = \begin{cases} -\tan^{-1} \frac{1}{u} & + \text{a constant.} \\ \tan^{-1} \frac{u+c}{1-cu} & + \text{a constant.} \end{cases}$$

$$269. \quad \log(x \pm yi) = \frac{1}{2} \log(x^2 + y^2) \pm i \tan^{-1} \frac{y}{x}.$$

$$270. \quad \sin^{-1} u = \cos^{-1} \sqrt{1-u^2} = \tan^{-1} \frac{u}{\sqrt{1-u^2}} = \csc^{-1} \frac{1}{u}.$$

$$271. \quad \cos^{-1} u = \sin^{-1} \sqrt{1-u^2} = \tan^{-1} \sqrt{\frac{1}{u^2} - 1} = \sec^{-1} \frac{1}{u}.$$

$$272. \quad \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left(\frac{x \pm y}{1 \mp xy} \right).$$

$$273. \quad \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} \pm y \sqrt{1-x^2}).$$

$$274. \quad \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} (xy \mp \sqrt{(1-x^2)(1-y^2)}).$$

$$275. \quad \sin x = \frac{e^{xi} - e^{-xi}}{2i}.$$

$$276. \quad \cos x = \frac{e^{xi} + e^{-xi}}{2}.$$

$$277. \quad \sin xi = \frac{1}{2} i (e^x - e^{-x}) = i \sinh x.$$

$$278. \quad \cos xi = \frac{1}{2} (e^x + e^{-x}) = \cosh x.$$

$$279. \quad \log_e x = (2.3025851) \log_{10} x.$$

The Natural Logarithms of Numbers between 1.0 and 9.9.

N.	0	1	2	3	4	5	6	7	8	9
1.	0.000	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2.	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3.	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.308	1.335	1.361
4.	1.386	1.411	1.435	1.459	1.482	1.504	1.526	1.548	1.569	1.589
5.	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6.	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7.	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8.	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9.	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

The Natural Logarithms of Whole Numbers from 10 to 109.

N.	0	1	2	3	4	5	6	7	8	9
1	2.303	2.398	2.485	2.565	2.639	2.708	2.773	2.833	2.890	2.944
2	2.996	3.045	3.091	3.135	3.178	3.219	3.258	3.296	3.332	3.367
3	3.401	3.434	3.466	3.497	3.526	3.555	3.584	3.611	3.638	3.664
4	3.689	3.714	3.738	3.761	3.784	3.807	3.829	3.850	3.871	3.892
5	3.912	3.932	3.951	3.970	3.989	4.007	4.025	4.043	4.060	4.078
6	4.094	4.111	4.127	4.143	4.159	4.174	4.190	4.205	4.220	4.234
7	4.248	4.263	4.277	4.290	4.304	4.317	4.331	4.344	4.357	4.369
8	4.382	4.394	4.407	4.419	4.431	4.443	4.454	4.466	4.477	4.489
9	4.500	4.511	4.522	4.533	4.543	4.554	4.564	4.575	4.585	4.595
10	4.605	4.615	4.625	4.635	4.644	4.654	4.663	4.673	4.682	4.691

The Values in Circular Measure of Angles which are given in Degrees and Minutes.

1'	0.0003	9'	0.0026	3°	0.0524	20°	0.3491	100°	1.7453
2'	0.0006	10'	0.0029	4°	0.0698	30°	0.5236	110°	1.9199
3'	0.0009	20'	0.0058	5°	0.0873	40°	0.6981	120°	2.0944
4'	0.0012	30'	0.0087	6°	0.1047	50°	0.8727	130°	2.2689
5'	0.0015	40'	0.0116	7°	0.1222	60°	1.0472	140°	2.4435
6'	0.0017	50'	0.0145	8°	0.1396	70°	1.2217	150°	2.6180
7'	0.0020	1°	0.0175	9°	0.1571	80°	1.3963	160°	2.7925
8'	0.0023	2°	0.0349	10°	0.1745	90°	1.5708	170°	2.9671

NATURAL TRIGONOMETRIC FUNCTIONS.

Angle.	Sin.	Csc.	Tan.	Ctn.	Sec.	Cos.	
0°	0.000	∞	0.000	∞	1.000	1.000	90°
1	0.017	57.30	0.017	57.29	1.000	1.000	89
2	0.035	28.65	0.035	28.64	1.001	0.999	88
3	0.052	19.11	0.052	19.08	1.001	0.999	87
4	0.070	14.34	0.070	14.30	1.002	0.998	86
5°	0.087	11.47	0.087	11.43	1.004	0.996	85°
6	0.105	9.567	0.105	9.514	1.006	0.995	84
7	0.122	8.206	0.123	8.144	1.008	0.993	83
8	0.139	7.185	0.141	7.115	1.010	0.990	82
9	0.156	6.392	0.158	6.314	1.012	0.988	81
10°	0.174	5.759	0.176	5.671	1.015	0.985	80°
11	0.191	5.241	0.194	5.145	1.019	0.982	79
12	0.208	4.810	0.213	4.705	1.022	0.978	78
13	0.225	4.445	0.231	4.331	1.026	0.974	77
14	0.242	4.134	0.249	4.011	1.031	0.970	76
15°	0.259	3.864	0.268	3.732	1.035	0.966	75°
16	0.276	3.628	0.287	3.487	1.040	0.961	74
17	0.292	3.420	0.306	3.271	1.046	0.956	73
18	0.309	3.236	0.325	3.078	1.051	0.951	72
19	0.326	3.072	0.344	2.904	1.058	0.946	71
20°	0.342	2.924	0.364	2.747	1.064	0.940	70°
21	0.358	2.790	0.384	2.605	1.071	0.934	69
22	0.375	2.669	0.404	2.475	1.079	0.927	68
23	0.391	2.559	0.424	2.356	1.086	0.921	67
24	0.407	2.459	0.445	2.246	1.095	0.914	66
25°	0.423	2.366	0.466	2.145	1.103	0.906	65°
26	0.438	2.281	0.488	2.050	1.113	0.899	64
27	0.454	2.203	0.510	1.963	1.122	0.891	63
28	0.469	2.130	0.532	1.881	1.133	0.883	62
29	0.485	2.063	0.554	1.804	1.143	0.875	61
30°	0.500	2.000	0.577	1.732	1.155	0.866	60°
31	0.515	1.942	0.601	1.664	1.167	0.857	59
32	0.530	1.887	0.625	1.600	1.179	0.848	58
33	0.545	1.836	0.649	1.540	1.192	0.839	57
34	0.559	1.788	0.675	1.483	1.206	0.829	56
35°	0.574	1.743	0.700	1.428	1.221	0.819	55°
36	0.588	1.701	0.727	1.376	1.236	0.809	54
37	0.602	1.662	0.754	1.327	1.252	0.799	53
38	0.616	1.624	0.781	1.280	1.269	0.788	52
39	0.629	1.589	0.810	1.235	1.287	0.777	51
40°	0.643	1.556	0.839	1.192	1.305	0.766	50°
41	0.656	1.524	0.869	1.150	1.325	0.755	49
42	0.669	1.494	0.900	1.111	1.346	0.743	48
43	0.682	1.466	0.933	1.072	1.367	0.731	47
44	0.695	1.440	0.966	1.036	1.390	0.719	46
45°	0.707	1.414	1.000	1.000	1.414	0.707	45°
	Cos.	Sec.	Ctn.	Tan.	Csc.	Sin.	Angle.

Values of the Complete Elliptic Integrals, K and E , for Different
Values of the Modulus, k .

$\sin^{-1}k$	K	E	$\sin^{-1}k$	K	E	$\sin^{-1}k$	K	E
0°	1.5708	1.5708	30°	1.6858	1.4675	60°	2.1565	1.2111
1°	1.5709	1.5707	31°	1.6941	1.4608	61°	2.1842	1.2015
2°	1.5713	1.5703	32°	1.7028	1.4539	62°	2.2132	1.1920
3°	1.5719	1.5697	33°	1.7119	1.4469	63°	2.2435	1.1826
4°	1.5727	1.5689	34°	1.7214	1.4397	64°	2.2754	1.1732
5°	1.5738	1.5678	35°	1.7312	1.4223	65°	2.3088	1.1638
6°	1.5711	1.5665	36°	1.7415	1.4248	66°	2.3439	1.1545
7°	1.5767	1.5649	37°	1.7522	1.4171	67°	2.3809	1.1453
8°	1.5785	1.5632	38°	1.7633	1.4092	68°	2.4198	1.1362
9°	1.5805	1.5611	39°	1.7748	1.4013	69°	2.4610	1.1272
10°	1.5828	1.5589	40°	1.7868	1.3931	70°	2.5046	1.1184
11°	1.5854	1.5564	41°	1.7992	1.3849	71°	2.5507	1.1096
12°	1.5882	1.5537	42°	1.8122	1.3765	72°	2.5998	1.1011
13°	1.5913	1.5507	43°	1.8256	1.3680	73°	2.6521	1.0927
14°	1.5946	1.5476	44°	1.8396	1.3594	74°	2.7081	1.0844
15°	1.5981	1.5442	45°	1.8541	1.3506	75°	2.7681	1.0764
16°	1.6020	1.5405	46°	1.8691	1.3418	76°	2.8327	1.0686
17°	1.6061	1.5367	47°	1.8848	1.3329	77°	2.9026	1.0611
18°	1.6105	1.5326	48°	1.9011	1.3238	78°	2.9786	1.0538
19°	1.6151	1.5283	49°	1.9180	1.3147	79°	3.0617	1.0468
20°	1.6200	1.5238	50°	1.9356	1.3055	80°	3.1534	1.0401
21°	1.6252	1.5191	51°	1.9539	1.2963	81°	3.2553	1.0338
22°	1.6307	1.5141	52°	1.9729	1.2870	82°	3.3699	1.0278
23°	1.6365	1.5090	53°	1.9927	1.2776	83°	3.5004	1.0223
24°	1.6426	1.5037	54°	2.0133	1.2681	84°	3.6519	1.0172
25°	1.6490	1.4981	55°	2.0347	1.2587	85°	3.8317	1.0127
26°	1.6557	1.4924	56°	2.0571	1.2492	86°	4.0528	1.0086
27°	1.6627	1.4864	57°	2.0804	1.2397	87°	4.3387	1.0053
28°	1.6701	1.4803	58°	2.1047	1.2301	88°	4.7427	1.0026
29°	1.6777	1.4740	59°	2.1300	1.2206	89°	5.4349	1.0008

The Common Logarithms of $\Gamma(n)$ for Values of n between 1 and 2.

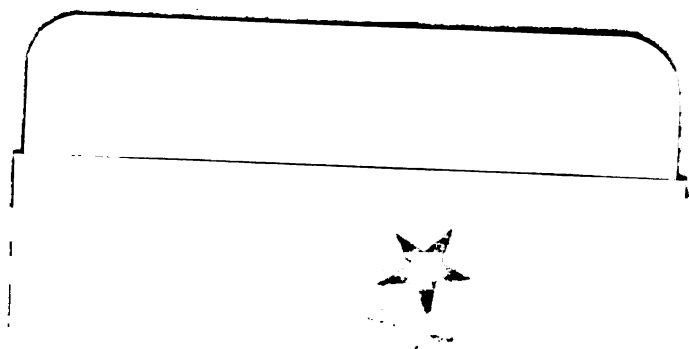
n	$\log_{10} \Gamma(n)$	n	$\log_{10} \Gamma(n)$	n	$\log_{10} \Gamma(n)$	n	$\log_{10} \Gamma(n)$	n	$\log_{10} \Gamma(n)$
1.01	1.9975	1.21	1.9617	1.41	1.9478	1.61	1.9517	1.81	1.9704
1.02	1.9951	1.22	1.9605	1.42	1.9476	1.62	1.9523	1.82	1.9717
1.03	1.9928	1.23	1.9594	1.43	1.9475	1.63	1.9529	1.83	1.9730
1.04	1.9905	1.24	1.9583	1.44	1.9473	1.64	1.9536	1.84	1.9743
1.05	1.9883	1.25	1.9573	1.45	1.9473	1.65	1.9543	1.85	1.9757
1.06	1.9862	1.26	1.9564	1.46	1.9472	1.66	1.9550	1.86	1.9771
1.07	1.9841	1.27	1.9554	1.47	1.9473	1.67	1.9558	1.87	1.9786
1.08	1.9821	1.28	1.9546	1.48	1.9473	1.68	1.9566	1.88	1.9800
1.09	1.9802	1.29	1.9538	1.49	1.9474	1.69	1.9575	1.89	1.9815
1.10	1.9783	1.30	1.9530	1.50	1.9475	1.70	1.9584	1.90	1.9831
1.11	1.9765	1.31	1.9523	1.51	1.9477	1.71	1.9593	1.91	1.9846
1.12	1.9748	1.32	1.9516	1.52	1.9479	1.72	1.9603	1.92	1.9862
1.13	1.9731	1.33	1.9510	1.53	1.9482	1.73	1.9613	1.93	1.9878
1.14	1.9715	1.34	1.9505	1.54	1.9485	1.74	1.9623	1.94	1.9895
1.15	1.9699	1.35	1.9500	1.55	1.9488	1.75	1.9633	1.95	1.9912
1.16	1.9684	1.36	1.9495	1.56	1.9492	1.76	1.9644	1.96	1.9929
1.17	1.9669	1.37	1.9491	1.57	1.9496	1.77	1.9656	1.97	1.9946
1.18	1.9655	1.38	1.9487	1.58	1.9501	1.78	1.9667	1.98	1.9964
1.19	1.9642	1.39	1.9483	1.59	1.9506	1.79	1.9679	1.99	1.9982
1.20	1.9629	1.40	1.9481	1.60	1.9511	1.80	1.9691	2.00	0.0000

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